Extract of a letter from the elder Mr. Euler to Mr. Bernoulli, concerning the Memoir published among those of 1771

Having read with much pleasure your investigations on numbers of the form $10^p \pm 1$, I have the honor of communicating to you the criteria by which one can judge, for each prime number $2p + 1$, which of the two formulas $10^p + 1$ or $10^p - 1$ will be divisible by $2p + 1$.

For this purpose, it is necessary to distinguish the following two cases.

First Case. If $2p + 1 = 4n + 1$, one has only to consider the divisors of the three numbers n, $n - 2$, and $n - 6$, and if among them one finds either both the numbers 2 and 5, or neither of them, that indicates that the formula $10^p - 1$ will be divisible; but if among the said divisors only one of the numbers 2 or 5 is found, then the formula $10^p + 1$ will be divisible. Thus, for the prime number $2p + 1 = 53 = 4n + 1$, we will have $n = 13$, and our three numbers will be 13, 11, 7, then neither 2 nor 5 is a divisor, and therefore the formula $10^{26} - 1$ will be divisible by 53.

Second Case. If $2p + 1 = 4n - 1$, one must consider these three numbers n, $n + 2$, and $n + 6$, and if among their divisors either both the numbers 2 and 5 are encountered, or neither of them, then the formula $10^p - 1$ will be divisible; but if only one of the numbers 2 and 5 is found to be among them, then the formula $10^p + 1$ will be divisible. For example if $2p + 1 = 59 = 4n - 1$, and therefore $n = 15$, our three numbers are 15, 17, 21, where 5 is among the divisors but not 2, so the formula $10^{29} + 1$ will be divisible by 59.

These rules are based on a principle whose proof is not yet known.

The largest prime number that we know is without doubt $2^{31} - 1 = 2147483647$, which Fermat has already verified to be prime; and I have also proved it; for, since this formula can admit divisors only of the two forms $248n + 1$ and $248n + 63$, I have examined all the prime numbers contained in these two formulas until 46339, none of which was found to be a divisor.

This progression

$$41,\ 43,\ 47,\ 53,\ 61,\ 71,\ 83,\ 97,\ 113,\ 131,\ \text{etc.}$$

whose general term is

$$41 - x + xx,$$

is all the more remarkable because the first forty terms are all prime numbers.

Extrait d'une lettre de M. Euler le père à M. Bernoulli, concernant le Mémoire imprimé parmi ceux de 1771. p. 318, Nouveaux mémoires de l'académie des sciences de Berlin (1774), 35–36. Number 461 in the Eneström index. Translated by Todd Doucet.